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TOPICAL ISSUE

Analysis of Peak Effects in the Solutions of a Class of Difference Equations

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Abstract—We consider a class of well-known high-order trinomial linear difference equations and analyze the non-asymptotic behavior of their solutions under non-zero initial conditions from the unit box. It is shown that, for certain subsets of coefficients in the stability domain, there always exist initial conditions leading to *peak*, a large deviation of solutions from the equilibrium position, and that these deviations may take arbitrarily large values. Various special cases are studied, numerical examples are presented.

Keywords: trinomial difference equation, stability, nonzero initial conditions, peak of solutions

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1. INTRODUCTION

Since relatively recently, there is an interest in studying and understanding the phenomenon of *peak of solutions* of stable differential or difference equations caused by the presence of nonzero initial conditions in the absence of exogenous disturbances. By peak we mean possible large deviations of solutions from the nonzero initial conditions at finite time intervals. The importance of this line of research is stipulated by many reasons. Obviously, such effects are highly undesirable in the engineering practice [1, 2] and may lead to malfunctioning or failure of digital control systems. Next, if a linearized model is adopted as a simplified description of a real-life nonlinear system around an equilibrium point, then, due to possible large deviations, the trajectory may leave the domain of attraction of the original nonlinear system so that the linear model is no more valid. Finally, the convergence of some powerful modern methods of minimization may happen to be non-monotonic, and this phenomenon is to be explained; see [3] for a very recent publication.

About ten years ago, Boris Theodorovich Polyak took a keen interest in this line of research, and since then he has written many papers on the subject; among the most important ones we mention [3–8], which are devoted to the analysis of the peak phenomenon both in continuous time (differential equations) and discrete time (difference equations). With this interest Boris Theodorovich inspired many of his students and followers, including the author of the present paper, where the peak effect is analyzed for a class of difference equations.

Whereas the *continuous-time* case is somewhat explored (e.g., see paper [5] with extended bibliography, including the cornerstone paper [9]), very little attention has been paid to *difference equations*. There are just scattered results in the literature, related to many-dimensional discrete-time systems and to numerical construction of upper bounds on peaks [10, 11]; adaptive control problems [12]; dependence of peaks on the degree of controllability of the system [13]. However, to the best of our knowledge, the simplest and most natural formulations of the problem, such as the evaluation of peaks in scalar difference equations, have not been considered. Perhaps the first work in this direction is [6], where the exact closed-form expression for peaks and lower bounds were obtained for several root locations of the characteristic polynomial and various initial conditions; certain specific equations were analyzed; equations with nonzero deterministic noise were studied. The exploited theory and methods of difference equations are given in [14].

Yet another interesting direction of research, the development of a probabilistic approach to the evaluation of peak is worth mentioning. Within this approach, the coefficients and/or initial conditions of a stable difference equation are assumed to be random, and an attempt is made to estimate the probability of the presence of peak, its mathematical expectation, etc.; see [8, 15].

In this paper, we analyze peak effects as applied to a class of trinomial equations specific to population dynamics problems using linearized models. The first notable paper studying this equation is [16] (to date, this paper has 257 citations in the Google Scholar bibliographic database), where necessary and sufficient conditions for its asymptotic stability were obtained in terms of its the coefficients; namely, an explicit description of the stability domain on the plane of the two coefficients was given.

These results were later generalized toward the presence of multiple delays [17, 18], complex-valued coefficients [19], vector equations [20, 21], finding alternative proofs for the shape of the stability domain [18], applications to continuous systems with delays [22], etc.; we also note [14], where instructive discussions are given.

A nice feature of this equation is that, having just three terms, it may exhibit nontrivial and diverse behavior for various values of the coefficients and initial conditions. Also, since just two coefficients are involved, the analysis of solutions is easier to perform in terms of the coefficients, not the roots.

The first results on possible peaks in this equation were obtained in [6]; here, we continue this line of research and find exact formulae for the magnitude of peak and peak instant. We show that, in certain situations, peak is unavoidable; moreover, its magnitude, as well as the peak instant may take arbitrarily large values.

2. NOTATION, DEFINITIONS, STATEMENT OF THE PROBLEM

In what follows, the standard notaion is used: $\mathbb{R}^n$ is the field of real numbers; the sign $\gg$ means "much greater than"; the symbol $\approx$ means "is approximately equal to"; the symbol $:=$ corresponds to "denote by"; the symbols $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote rounding to the nearest integer toward negative/positive infinity; $|\cdot|$ is the absolute value of a number; λ_i is the ith root of a polynomial; e is the base of the natural logarithm function; for integers $s \geqslant t \geqslant 0$, the binomial coefficient is denoted by $\binom{s}{t}$.

We consider the following scalar, linear homogenous trinomial equation of order $n+1$:

$$x_{k+1} - a x_k + b x_{k-n} = 0, \qquad k = 1, 2, \ldots \tag{1}$$

with real coefficients a, b and characteristic polynomial

$$p(\lambda) = \lambda^{n+1} - a\lambda^n + b.$$

The initial conditions $x^{(0)} = (x_{-n}, \ldots, x_{-1}, x_0) \in \mathbb{R}^{n+1}$ are assumed to have the unit norm: $\|x^{(0)}\|_\infty = 1$. This is without loss of generality, since the solution is a linear function in $x^{(0)}$.

Of our interest are only asymptotically stable equations (1); i.e. those having roots λ_i inside the unit disk on the complex plane, $|\lambda_i| < 1$. Denote by $\mathcal{S}_n \subset \mathbb{R}^2$ the stability domain of equation (1) on the coefficient plane; obviously, it is nonempty. For $(a, b) \in \mathcal{S}_n$, the solutions assymtotically tend to zero; we are interested in the analysis of behavior of solutions for finite values of k.

For given $(a,b)\in\mathcal{S}_n$ and $x^{(0)}$, let

$$\varkappa(a,b,x^{(0)})=\max_{k\geqslant 1}|x_k|$$

denote the maximum absolute value of the solution (trajectory). We omit the arguments in $\varkappa(a,b,x^{(0)})$ unless it leads to any ambiguity. The solution is said to *experience peak* if this quantity is greater than unity, and the associated number of iteration $k^*=\arg\max_k|x_k|$ is referred to as the *peak instant.*

We will also try to characterize the *peak domain*, a part $\mathcal{P}_n$ of the stability domain $\mathcal{S}_n$, such that for every point in this domain peak is observed for at least one initial condition. Of interest are also peak domains associated with specific initial conditions.

The linearized model (1) of population dynamics was introduced in [23]; the quantities involved have the following physical meaning: k is the year of observation of the populaion, x_k is change of size of the population in year k, n is the fertility age, a is the survival coefficient, b is the recruitment of the pulation (birth rate). The initial conditions correspond to the size of the population in the n years preceding the start of monitoring the population using this model. The stability of the equation is synonimous to the invariance of the population size in time, whereas peak of solutions means an abnormally large current increase or decline. Both are undesirable from the point of view of ecological equilibrium.

3. MAIN RESULTS

3.1. Stability Domain and Peak Domain on the Plane of Coefficients of equation (1)

Figure 1 depicts the stability domain of equation (1) on the coefficient plane; see [16, 18].

It is easy to show that, for the coefficients in the *Cohn domain* $\mathcal{C}=\{a,b\colon |a|+|b|<1\}$ (dotted line) there is no peak, no matter what the initial conditions are. We are only intrested in domains composed of the sets I and II ("winglets"). For every pont (a,b) in these sets we have $|a|+|b|\geqslant 1$, and peak-promoting initial conditions do exist; see Theorem 1 in [15].

A very simple and accurate upper bound on the the area of the peak domain $\mathcal{P}_n$ can be derived. Indeed, from the eqution of the boundary of $\mathcal{S}_n$ (see [16]) the right winglet is seen to be located in the triangle with the vertices $(0,1)$, $(1+1/n,1/n)$, and $(1,0)$ and having area $1/n$. The corresponding left triangle has the same area, and since $\mathbf{Vol}(\mathcal{C})=2$, we obtain

$$\mathbf{Vol}(\mathcal{P}_n)<\frac{1}{n+1}\mathbf{Vol}(\mathcal{S}_n)$$

for $n>1$ (for $n=1$ we have equality in the estimate above). By symmetry (see Fig. 1) this also holds true for even vlues of n. Therefore, the "probability" for equation (1) to experience

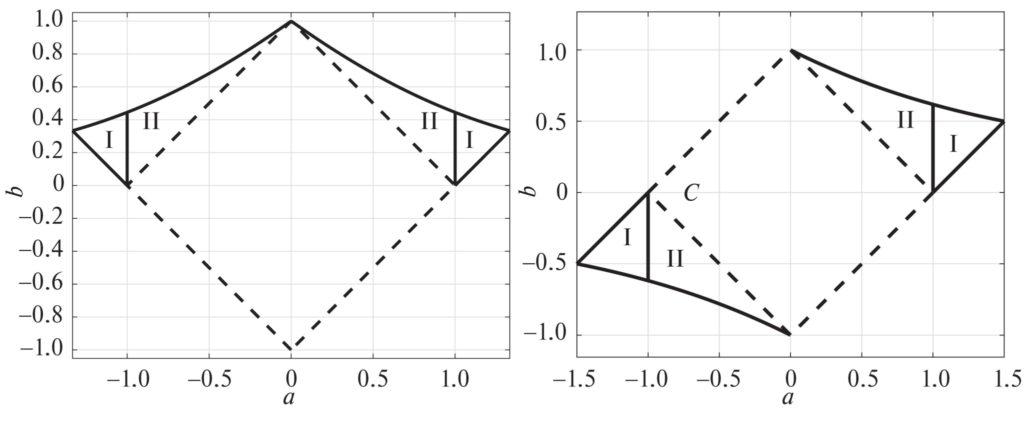

Fig. 1. The stability fomain S_n of equation (1) for $n=2$ (left) and $n=3$ (right).

peak decreases as the order of equation grows (for instance, already for $n = 7$ we have $\mathbf{Vol}(\mathcal{P}_n) \approx 0.08 \times \mathbf{Vol}(\mathcal{S}_n)$); however, in the sequel we will show that the *peak magnitude* can take arbitrarily large values. Note that, for equations of general type, the situation is quite the opposite; i.e., as the order of equation grows, peak effects are a typical phenomena; see [15].

3.2. Magnitude of Peak under Canonical Initial Conditions

We now turn to the evaluation of the peak magnitude in equation (1). For known degree n, coefficients $(a, b) \in \mathcal{P}_n$, and initial conditions $x^{(0)}$, the solution can be found numerically by iterating over k. However, we are interested in closed-form estimates of the peak values. In general, this is not doable, and below we provide estimates of the magnitude of peak and peak instant for specific initial conditions and certain sets of the coefficients.

In what follows, we analyze the nonasymptotic behavior of solutios under "canonical" initial conditions

$$x^{(0)} = (0, \ldots, 0, 1). \tag{2}$$

3.2.1. A simple lower bound. With $x^{(0)} = (0, \ldots, 0, 1)$, equation (1) immediately implies that the first n iterations x_k do not depend on b:

$$x_k = a^k, \qquad k = 1, 2, \ldots, n, \tag{3}$$

meaning that peak is observed for all $(a, b) \in \mathcal{S}_n$, $|a| > 1$; in other words, the domains I are peak domains for $x^{(0)} = (0, \ldots, 0, 1)$. Hence, the magnitude of peak can be evaluated from below by

$$\varkappa \geqslant \varkappa_* = |a|^n > 1. \tag{4}$$

However, clearly, this estimate may happen to be rather poor.

Example 1. For $n = 10$ and $\varepsilon = 0.001$, consider the coefficients $a = \frac{n+1}{n} - \varepsilon = 1.099$ and $b = \frac{1}{n} = 0.1$. We have $\varkappa_* = 2.5703$, whereas the true value of the peak magnitude obtined by direct computations is equal to $\varkappa = 13.0732$, and it is attained at $k = 111 \gg n$.

3.2.2. An exact conbinatorial formula. So, for the initial conditions $x^{(0)}$ as in (2), we have

$$x_k = a^k := X_{k,0}, \quad 0 \leqslant k \leqslant n.$$

At the next $n + 1$ steps, the value of x_k depends on b, and straightforward though lengthy computations yield

$$\begin{aligned} x_k &= X_{k,0} - \binom{k-n}{1} a^{k-(n+1)} b \\ &\doteq X_{k,0} - X_{k,1} \\ &\qquad \text{for } \ n+1 \leqslant k \leqslant 2(n+1) - 1, \end{aligned}$$

where the b-dependent term $\binom{k-n}{1} a^{k-(n+1)} b$ is denoted by $X_{k,1}$.

Over the third $(n + 1)$-tuple of iterations, the solution depends also on b^2:

$$\begin{aligned} x_k &= X_{k,0} - X_{k,1} + \binom{k-2n}{2} a^{k-2(n+1)} b^2 \\ &\doteq X_{k,0} - X_{k,1} + X_{k,2} \\ &\qquad \text{for } \ 2(n+1) \leqslant k \leqslant 3(n+1) - 1, \end{aligned}$$

where the b^2-dependent term $\binom{k-2n}{2} a^{k-2(n+1)} b^2$ is denoted by $X_{k,2}$, and so forth.

By iterating in k and collecting the formulae above, we obtain the following result.

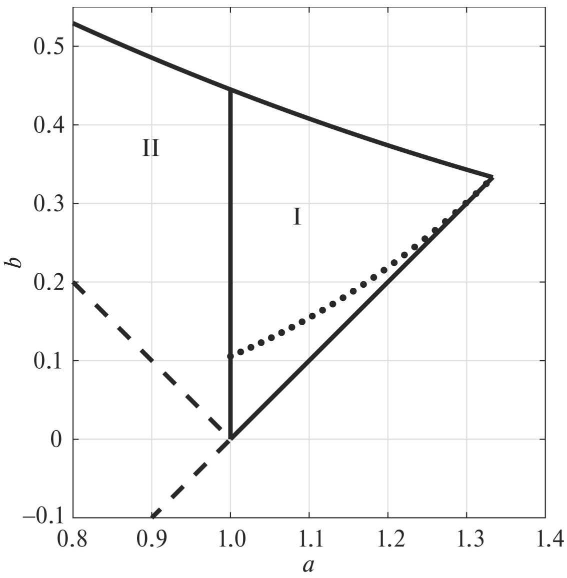

Fig. 2. The parametric family (6) of the coefficients of equation (1).

Assertion 1. *The solution* x_k *of equation* (1) *under initial conditions* (2) *has the following form*:

$$x_k = \sum_{j=0}^{\lfloor k/(n+1) \rfloor} (-1)^j \binom{k-jn}{j} a^{k-j(n+1)} b^j, \qquad k = 0, 1, \ldots . \tag{5}$$

The formula above presents the solution of (1), (2) in closed form; however, it can hardly be used to evaluate the peak magnitude and instant.

3.2.3. A parametric family of equations (1)**.** Following [6], consider the family of coefficients

$$a = 1 + \frac{\alpha}{n}, \quad 0 < \alpha < 1, \quad b = a^{n+1} \frac{n^n}{(n+1)^{n+1}}, \tag{6}$$

where α is a parameter. It was shown in [6] that the maximum in absolute value root of equation (1), (6) is equal to

$$\rho = \frac{an}{n+1} = \frac{n+\alpha}{n+1}$$

and it has multiplicity two.

From the equation of the boundary of $\mathcal{S}_n$ (e.g., see [14], Theorem 5.3) it follows that the point (a, b) (6) belongs to the set I (part of the right winglet; the anlysis of the left winglet is the same due to symmetry). This parametric family is depicted by the dotted line in Fig. 2.

The behavior of solutions of this equation is easier to analyse as compared to the general case; in [6] special type of initial conditions were considered, closed-form solutions were obtained for equation (1), and explicit formulae for the peak magnitude and peak instant were derived. Here we consider the canonical initial conditions (2).

Let us first analyze the simplest case $n = 1$; i.e., the second-order equation. From (6) we have

$$a = 1 + \alpha, \quad 0 < \alpha < 1, \qquad b = a^2/4,$$

so that the roots of the characteristic equation are equal to $\lambda_1 = \lambda_2 = \rho = \frac{a}{2} < 1$. For the initial conditions $x_{-1} = 0$, $x_0 = 1$ we immediately obtain

$$x_k = (k+1)\rho^k, \quad k = 1, 2, \ldots \tag{7}$$

and the peak magnitude is easy to evaluate. For the peak instant we have

$$k^* = \max\{k\colon x_{k-1} < x_k\}$$

and by differentiation in k we arrive at

$$k^* = \left\lfloor \frac{\rho}{1-\rho} \right\rfloor = \left\lfloor \frac{1+\alpha}{1-\alpha} \right\rfloor. \tag{8}$$

Substitution of $k = k^*$ in (7) leads to the following lower bound for the peak magnitude:

$$\varkappa > \Big(\frac{\rho}{1-\rho} + 1\Big)\rho^{\frac{\rho}{1-\rho}} > \frac{2}{(1-\alpha)\mathrm{e}}.$$

Therefore, these two formulae show that even for the second-order equation both the peak instant and the peak magnitude can take arbitrarily large values as $\alpha \to 1$; i.e., as the coefficients approach the boundary of the stability domain.

3.2.4. More on conservatism of estimate (4)**.** We now consider the general case $n > 1$. As we have already mentioned, over the first n iterations the solution has the form (3). It increases monotonically for any feasible a, and $\varkappa_* = a^n$ can be adopted as a lower bound on peak. It can be shown that, for the initial conditions (2) and a, b of the form (6), the solution x_k is unimodal, and the peak value is *equal* to $x_n = a^n$ if and only if $x_n > x_{n+1} = a^{n+1} - b$. With account to the expression (6) for b we see that for

$$1 < \alpha \leqslant \alpha_1 = \frac{n^{n+1}}{(n+1)^{n+1} - n^n} \approx \frac{1}{\mathrm{e}},$$

the peak value is given by $\varkappa_1 = (1 + \alpha/n)^n \approx \mathrm{e}^\alpha$, and for $\alpha > \alpha_1$ it takes greater values.

Example 2. For $n = 10$ we have $\alpha_1 = 0.3232$, and $\varkappa_* = 1.3745$ gives the exact value of the peak magnitude for all $\alpha \leqslant \alpha_1$. However for $\alpha = 0.9$, the true value of peak is equal to $\varkappa = 7.5965 \gg \varkappa_* = a^n = 2.3674$, and it is attained at step $k = 106 \gg n$. For comparison, the magnitude of peak under (presumably worst-case) initial conditions $x^{(0)} = (-1, \dots, -1, 1)$ is equal to $\varkappa = 14.4601$.

Hence, as for the coefficients a, b of the general form, the estimate $\varkappa_*$ of the peak magnitude may happen to be very poor if $\alpha > \alpha_1$. Below, a much more accurate estimate will be obtained.

3.2.5. A closed-form lower bound. Having formula (7) in mind, by inducion in n we obtain the following result.

Assertion 2. *For any $k \geqslant 1$, the solution of* (1) *with coefficients* (6) *and initial conditions $x^{(0)} = (0 \dots, 0, 1)$ is bounded from below by*

$$x_k \geqslant y_k = \frac{2}{n+1}(k+1)\rho^k, \quad k = 1, 2, \dots. \tag{9}$$

For large values of k this estimate represents the asymptotics for the solutions of x_k.

The peak instant k_y^* for the sequence y_k can be adopted as an estimate of the true peak instant k^*:

$$k^* \approx k_y^* = \left\lfloor \frac{\rho}{1-\rho} \right\rfloor = \left\lfloor \frac{n+\alpha}{1-\alpha} \right\rfloor. \tag{10}$$

Respectively, by substituting the right-hand side of (10) into (9), we obtain the following estimate $\widehat{\varkappa}_y$ of the peak magnitude for x_k:

$$\varkappa \geqslant \widehat{\varkappa}_y \approx \frac{2}{(1-\alpha)\mathrm{e}}\,\frac{n+1}{n+\alpha}. \tag{11}$$

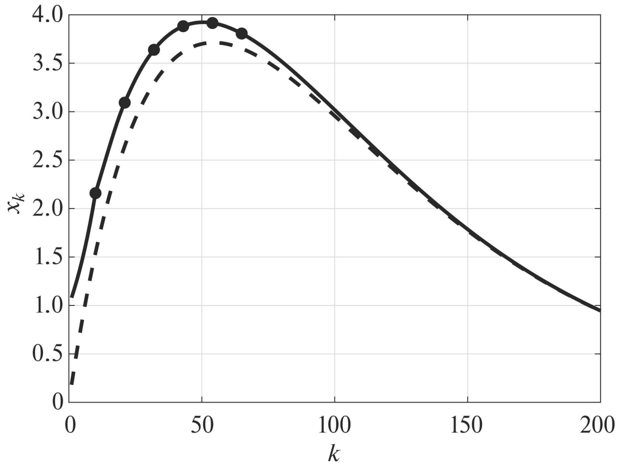

Fig. 3. The actual solution x_k (upper curve) for $n = 10$, $\alpha = 0.8$; the values at the endpoints of the first six $(n + 1)$-tuples (bold dots); the values of y_k (lower curve).

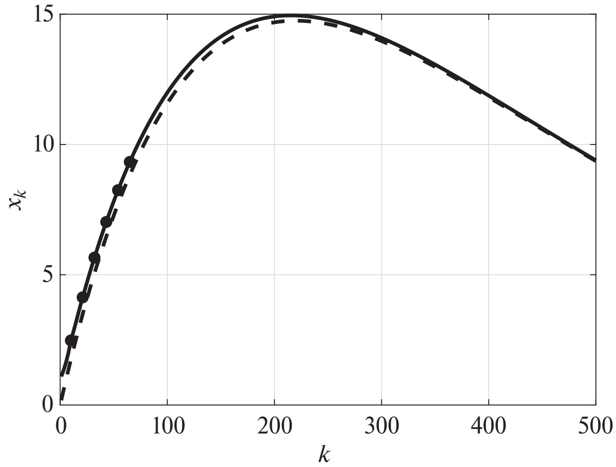

Fig. 4. Same as in Fig. 3, but for $\alpha = 0.95$.

An interesting observation can be made from formulae (10) and (11). Increase in the order n under fixed α leads to an increase in the value of the peak instant, whereas the change in the peak magnitude itself is very small. On the other hand, for n fixed, increase in α leads to an increase of both quantities. This picture differs from the one observed for equations of the general type, where both these quantities (as a rule) grow with increasing order of the equation and the coefficients approaching the boundary of the stability domain; see the results in [6].

Example 3. In Fig. 3, the upper curve corresponds to the solution x_k for $n = 10$, $\alpha = 0.8$ (also, bold dots indicate the values of the trajectory at the endpoints of $(n + 1)$-tuples mentioned in Section 3.2.2), and the lower curve to the estimate y_k.

The true value of peak is $\varkappa = 3.9227$, and it is attained at step $k^* = 51$. Estimates (11) and (10) give $\widehat{\varkappa}_y = 3.7469$ and $k^*_y = 54$; the relative error of the estimate is approximately 4.5%.

For the same value $\alpha = 0.8$ but for a higher order $n = 20$ of the equation, we obtain almost the same value $\varkappa = 3.9274$ for the peak magnitude, but essentially larger value $k^* = 97$ for the peak instant. The difference with the previous experiment is explained by expressions (11) and (10).

On the other hand, for the same $n = 10$ and a much higher value $\alpha = 0.95$, the estimates give $\widehat{\varkappa}_y = 14.7824$ and $k^*_y = 218$, whereas the true value of peak is equal to $\varkappa = 14.9517$, and it is attained at step $k^* = 216$, see Fig. 4. Both the peak magnitude and the peak instant grew up significantly; the relative error of the estimate y_k is slightly more than 1%.

Therefore, the accuracy of estimate (9) increases as the value of α approaches unity, which is exactly the case when peak is observed at distant iterations and the application of the general formula (5) requires some effort.

3.2.6. Yet another parametric family. We briefly discuss yet another parametric family of equations (1), which is defined by

$$a = 1 + \frac{1}{n} - \varepsilon, \quad b = \frac{1}{n}$$

with $0 < \varepsilon < \frac{1}{n}$ and initial conditions (2). For small ε, the feasible point (a, b) is located close to the right corner of the domain $\mathcal{S}_n$. As $\varepsilon \to 0$ and n fixed, the peak magnitude increases (since the maximum in absolute value root of equation (1) tends to $\rho = 1$ with multiplicity two), and this behavior is typical to equations of the general form; see [6].

Let us now fix a small vlue of ε and increase the order n of equation. In contrast to the setups considered in [6] (increase of the peak magnitude as the order increses), here we observe just the opposite phenomenon: The value of peak *decreases* as n grows, and for $n \geqslant \lceil \frac{1}{\varepsilon} \rceil$ there is no peak at all. A trivial explanation of this fact is that the point (a, b) approaches the point $(1, 0)$, which belongs to the no-peak domain $\mathcal{C}$.

3.2.7. Corner points of the domain $\mathcal{S}_n$. For certain specific coefficients $(a, b) \in \mathcal{S}_n$, the closed-form solution can be obtained and its behavior can be analyzed. For instance, these are the corner points of the stability domain.

The first "trivial" points are given by $a = 0$ and $b = \pm 1$ (the upper and lower corners in Fig. 1). For $b = 1$, the roots of the characteristic polynomial $p(\lambda) = \lambda^{n+1} + 1$ are roots of minus unity. They are evenly spaced on the circle of unit radius and we arrive at the result that immediately follows from Assertion 1 or it can be obtained by direct calculations.

Assertion 3. *For $a = 0$, $b = 1$ the solution of* (1), (2) *has the form*

$$x_k = \begin{cases} 0 & \text{for} \quad \operatorname{mod}(k, n) \neq 0, \\ (-1)^m & \text{for} \quad \operatorname{mod}(k, n) = 0. \end{cases}$$

For $b = -1$, the picture is pretty much the same, but the nonzero values of the solution are equal to plus unity.

A more interesting corner point is $a = \frac{n+1}{n}$, $b = \frac{1}{n}$ and its symmetric $a = -\frac{n+1}{n}$, $b = \frac{1}{n}$ (or $b = -\frac{1}{n}$), see Fig. 1; they both belong to the parametric family (6) with $\alpha = 1$. In that case, the largest root is $\rho = 1$ and it has multiplicity two. In contrast to the previous case, the solution x_k diverges, and from Assertion 2 it follows that its growth is at least linear in k. More accurately, we have the following result.

Assertion 4. *For $a = 1 + \frac{1}{n}$, $b = \frac{1}{n}$, the asymptotics of the solution of equation* (1), (2) *is given by*

$$x_k \sim \frac{2(k+1)}{n+1} + \frac{2}{3}\frac{n-1}{n+1} = \frac{2k}{n+1} + \frac{2}{3}\frac{n+2}{n+1}.$$

An illustration for $n = 10$ is presented in Fig. 5; the asymptotic formula is seen to be quite accurate even for small values of k, and the solution very quickly reaches the asymptotics.

3.2.8. An illustrative example. We finally present a numerical illustration of the magnitude of peak of solutions of equation (1) with initial conditions $x^{(0)} = (0, \dots, 0, 1)$ and arbitrary coefficients $(a, b) \in \mathcal{P}_n$.

In the simplest case $n = 1$, the peak domain is the triangle $\mathcal{P}_1^+ = \{1 < a < 2, a - 1 < b < 1\}$ on the plane of the coefficients and it symmetric $\mathcal{P}_1^- = \{-2 < a < 1, a + 1 < b < 1\}$. As shown above, peak takes place for all points in $\mathcal{P}_1^+$ and $\mathcal{P}_1^-$, and its minimum value is equal to $|a|$.

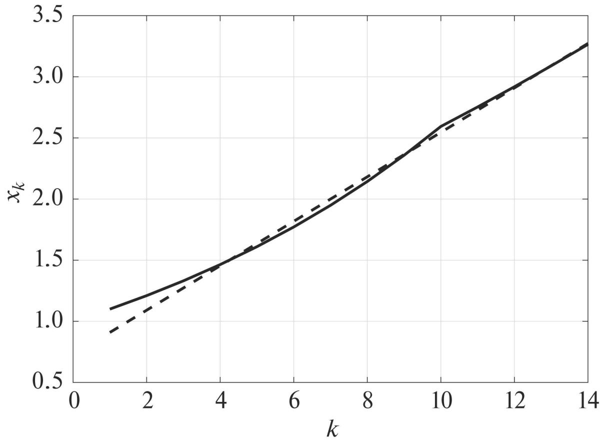

Fig. 5. The solution of equation (1), (2) for $a = 1 + \frac{1}{n}$, $b = \frac{1}{n}$, $n = 10$ (solid line) and its asymptotics (dotted line).

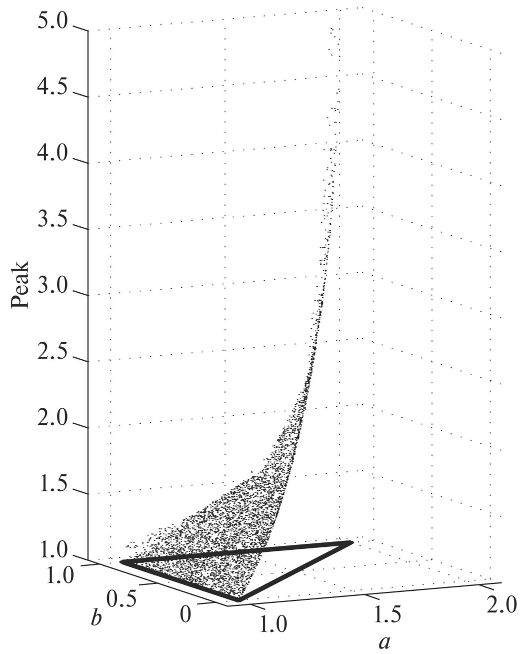

Fig. 6. The magnitude of peak at various points of the domain $\mathcal{P}_1^+$.

The following experiment was conducted. We sampled $N = 10\,000$ points randomly uniformly over the domain $\mathcal{P}_1^+$ and computed numerically the peak value for every equation with coefficients being the coordinates of the generated points. Figure 6 depicts the domain $\mathcal{P}_1^+$ (bold-line triangle) and the values of peaks of solutions of the corresponding equations. The domain of large values of peak is seen to be small and, in accordance with similar experiments, it gets smaller as the order n increases. However, in compliance with the results presented adove, the *peak magnitude* may take arbitrarily large values for the coefficients located closer the boundary of $\mathcal{P}_n^+$.

4. CONCLUSIONS

We analyzed peak effects of solutions of a well-known trinomial difference equation, which has a transparent pratical origin. It is shown that peak is inevitable if the coefficients belong to certain subsets of the stability region; also, it was demonstrated that the magnitude of peak and peak instant can take arbitrarily large values. Exact expressions for the magnitude of peak and peak instant or closed-form lower bounds are obtained for certain specific values of the coefficients and intial conditions.

Further research assumes the analysis of the nonasymptotic behavior of solutions of other families $(a, b) \in \mathcal{P}_n$; in particular, the ε-parameterized family from Section 3.2.6, as well as other special-type equations; e.g., those considered in [17]. Upper bounds on the value of peak are also worth paying attention for.

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